$\operatorname{Berry}_{\operatorname{Review}\operatorname{Project}}_{\operatorname{for}\operatorname{the}\operatorname{course}}\operatorname{Holonomy}$

CMPE 633C : Geometric Mechanics and Control

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2012-10-0105

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May 9, 2012

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1 Introduction

In this review project, we will first see how Berry Phase arises in a quantum system when it is subject to adiabatic changes through some closed path. We will then derive an explicit formula for Berry Phase and will see some of its properties. Then to see its connection with geometry, we will develop some basics of differential geometry from manifolds to tangent spaces to fibre bundles. Once we have the notion of a fibre bundle, we will define a connection on this fibre bundle. We will then see how the tangent space (of the total space of the Fibre Bundle) splits into a Vertical subspace and a Horizontal subspace. This will lead to the concept of a horizontal lift (of some curve in the base space to the total space). We will see that this naturally leads to the concept of Holonomy. After we have looked at Holonomy, we will see how we can see Berry Phase as a Holonomy. Finally, to understand the concept of Berry Phase as a Holonomy, we will look at two specific examples : A spin - $\frac{1}{2}$ particle in a magnetic field and the Ahoronov Bohm effect. We will see how Berry Phase arises in these examples and how can it be seen as a Holonomy.

2 Berry Phase

Berry Phase is the additional phase that arises along with the dynamical phase when a quantum system is subject to adiabatic changes through a closed path. To see how it arises, we first need to understand the quantum adiabatic theorem. Consider a time varying Hamiltonian $\hat{H}(t)$. It has non-degenerate eigenvalues $E_1, E_2, ..., E_n$ with eigenstates $|\phi_n(t)\rangle$ At time t = 0, the system is in the state $|\phi_n(0)\rangle$ The quantum adiabatic theorem states that if the variation in $\hat{H}(t)$ is slow, then at time t, the system is in a state $e^{i\gamma}|\phi_n(t)\rangle$ This is an instantaneous eigenstate of the Hamiltonian. In simpler terms, the system always remains in an eigenstate of the slowly varying Hamiltonian with at most a phase factor. This additional phase as we will see later cannot be ignored. This is the Berry Phase.

Now we will derive an explicit formula for Berry Phase.

Consider the parameter space of the Hamiltonian. It is described by R(t). Then, we can write the Hamiltonian as H = H(R(t)) This Hamiltonian is varied adiabatically and the system is transported around a closed path c in the parameter space such that R(0) = R(T)

Then, H(R(t)) has eigenstates $|n(R(t))\rangle$ such that

$$H(R)|n(R)\rangle = E_n(R)|n(R)\rangle \tag{2.1}$$

The evolution of the system is given by the Schrodinger's Equation,

$$\hat{H}(R)|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle$$
(2.2)

If the system is prepared in the state $|n(R(0))\rangle$, from the adiabatic theorem, it will be in a state $e^{i\gamma(t)}|n(R(t))\rangle$ after time t. The final state of the system can be written as:

$$|\psi(t)\rangle = e^{-i\theta(t)}e^{i\gamma_n(t)}|n(R(t))\rangle$$
(2.3)

where,

$$\theta(t) = \frac{1}{\hbar} \int_0^t dt \hat{E}_n(R(t)) \tag{2.4}$$

is the dynamical phase that comes whenever a system undergoes time evolution. $\gamma_n(t)$ is the geometrical phase called the Berry Phase and depends only on the geometry of the ciruit traversed in the parameter space.

We now use (2.4) and (2.3) in (2.2) to get,

$$\hat{H}|\psi\rangle = e^{-i\theta} e^{i\gamma_n(t)} E_n(R)|n(R)\rangle$$
(2.5)

and,

$$i\hbar\frac{d}{dt}|\psi\rangle = i\hbar[i\gamma_n(t)e^{-i\theta}e^{i\gamma_n(t)}|n(R)\rangle + e^{-i\theta}e^{i\gamma_n(t)}\frac{d}{dt}|n(R)\rangle - i\theta(t)e^{-i\theta}e^{i\gamma_n(t)}|n(R)\rangle$$
(2.6)

From the definition of θ , we get ,

$$i\hbar\dot{\theta} = iE_n(R(t)) \tag{2.7}$$

Using this in (2.5) and (2.6) we get,

$$\frac{d}{dt}|n(R)\rangle + i\dot{\gamma_n}|n(R)\rangle = 0$$
(2.8)

Taking its dot product with $\langle n(R)|,$ and with the normalization $\langle n(R)|n(R)\rangle=1$, we get,

$$\dot{\gamma_n} = i \langle n(R) | \frac{d}{dt} | n(R) \rangle \tag{2.9}$$

using the chain rule, we have $\frac{d}{dt} = \nabla_R \frac{dR}{dt}$ and integrating the above equation, we get,

$$\gamma_n = i \oint_c \langle n(R) | \nabla_R | n(R) \rangle dR \tag{2.10}$$

From this, it is clear that Berry Phase is only a geometrical property, it depends only on the ciruit traversed in the parameter space and not on how fast or slow it was traversed.

Now, we will look into some of its properties, Since $\langle n(R)|n(R)\rangle = 1$, we have,

$$\nabla_R \langle n(R) | n(R) \rangle = 0 \tag{2.11}$$

$$\Rightarrow \langle n(R) | \nabla_R | n(R) \rangle + (\langle n(R) | \nabla_R | n(R) \rangle)^* = 0$$
(2.12)

$$\Rightarrow Re(\langle n(R) | \nabla_R | n(R) \rangle) = 0 \tag{2.13}$$

 $\Rightarrow \gamma_n$ is purely Real.

Another important property of Berry Phase is that it is gauge invariant. define,

$$A_n(R) = i\langle n(R) | \nabla_R | n(R) \rangle \tag{2.14}$$

and consider a transformation of the states as,

$$|n(\tilde{R})\rangle = e^{i\zeta_n(R)}|n(R)\rangle \tag{2.15}$$

where $\zeta_n(R)$ is a single valued function on the parameter space. Then,

$$\tilde{A}_n = i \langle \tilde{n(R)} | \nabla_R | \tilde{n(R)} \rangle$$
(2.16)

$$\Rightarrow A_n(R) = iA_n(R) - \nabla_R \zeta_n(R)$$
(2.17)

$$\Rightarrow \tilde{\gamma_n} = \oint A_n(R) dR \tag{2.18}$$

Now since $\zeta_n(R)$ is single valued $\oint \nabla_R \zeta_n(R) dR = 0$

$$\Rightarrow \tilde{\gamma_n} = \gamma_n \tag{2.19}$$

Hence, the Berry Phase is gauge invariant.

Note that the equation (2.15) is just a transformation of basis. This means that we cannot remove this geometric phase by some transformation of the basis.

3 Geometric Interpretation of Berry Phase

Now that we have understood what a Berry Phase is, we will see its geometric interpretation. That is how can we understand it from a differential geometry perspective. Before we do that, we will revise very briefly some basics of differential geometry.

3.1Introduction to Differential Geometry

Differentiable Manifold

A differentiable manifold of dimension n is a set M and a family of injective mappings :

 $X_{\alpha}: U_{\alpha} \subseteq R_n \to M$ of open sets U_{α} such that :

i. $\bigcup_{\alpha} X_{\alpha}(U_{\alpha}) = M$

ii. For any pair α, β with $X_{\alpha}(U_{\alpha}) \bigcap X_{\beta}(U_{\beta}) = W \neq \phi$ the sets $X_{\alpha}^{-1}(W), X_{\beta}^{-1}(W)$ are open in \mathbb{R}^n and $X_{\beta}^{-1}oX_{\alpha}$ is differentiable. iii. The Family (X_{α}, U_{α}) is maximal w.r.t (i) and (ii).

In very simple terms, A Manifold is a set which locally "looks like" Euclidean space. Since we know how to do calculus on a Euclidean space, we cover all of the set by some mappings to Euclidean space. Then, whenever we want to compute something on the manifold, we simply use the maps and the inverse maps to bring the computation down to Euclidean space, perform the computation and then take it back to the Manifold. For example, R_n is a manifold, the sphere is a manifold (S^2) etc.

Tangent Vector

Let $\alpha: (-\epsilon, \epsilon) \to M$ be a curve in M with $\alpha(0) = P$ and parametrized by t. f is a differentiable function in the neighborhood of P. Then the tangent vector at t = 0 (or at P) is defined as:

 $\dot{\alpha}(0)f = \frac{d}{dt}(f \circ \alpha)|_{t=0}$. This can be written in terms of the local co-ordinates (which are the mappings from M to \mathbb{R}^n) as, $\dot{\alpha}(0)f = (\sum_{i=1}^n \dot{x_i}(0)\frac{\partial}{\partial x_i})f$

The set of all tangent vectors at P is called the **Tangent Space** denoted by $T_P M$

Differential Forms

The maps $\omega: T_P M \times T_P M \times \ldots T_P M \to R$ with the condition that, $\omega(v_1, v_2, \dots, v_k) = -\omega(v_2, v_1, \dots, v_k)$ are called k-forms on the manifold M

Push forward

consider a map $\phi: M \to N$ then, the push forward of this map is another map such that :

 $\phi_*: T_P M \to T_P N$

Pull back

The pull back (with respect to the map ϕ) of any form on the manifold is defined as,

 $\phi^*\omega(v_1, v_2, \dots, v_k) = \omega(\phi_*v_1, \phi_*v_2, \dots, \phi_*v_k)$

where v_1, v_2, \ldots, v_k belong to $T_P M$ and ω acts on vectors in $T_P N$

3.2 Fibre Bundles

Now, we will look at the most important concept needed to understand Berry Phase as a holonomy : The Fibre Bundle.

A fibre bundle is a space Q for which we have,

1. A base space B

2. A projection $\pi: Q \to B$

3. Fibres $\pi^{-1}(b)$ for $b \in B$. (All of these fibres are homeomorphic to F , This means that they are all homeomorphic to each other as well.)

4. A structure group G of homeomorphisms of F into itself. (This basically tells that how can we go from one fibre to another or on different points of the same fibre.)

We also have the conditions that:

i. B has a covering by open sets U_i such that Bundle is locally trivial i.e $\pi^{-1}(U_i)$ is homeomorphic to $U_i \times F$

ii. If h_j is the map giving homeomorphisms on fibres, then $h_j h_k^{-1} \in G$

So, we have a large space Q. Through the map π , it is projected onto a smaller space B. Since B is smaller than Q, a single point in B will correspond to multiple points in Q. The collection of all these points to which any single point b in B corresponds to is given by the inverse map $\pi^{-1}(b)$ and this is called the fibre.

Locally, the larger space Q looks like the product of these fibres and open sets in the samller space B but, globally, this is not necessarily true, Q can have totally different topologies. If it looks like $U_i \times F$ globally also, then it is called a **trivial fibre bundle**.

As an example consider a cylinder. The total space is the cylinder. The base space is a circle and the fibre is the real line. (The projection is that all of the real line is mapped to a single point on the circle) We can see it as the product of a circle and the real line. (i.e at each point of a circle, attach a real line and we get a cylinder.), So we can write the cylinder as $S^1 \times R$. Therefore, this is a trivial fibre bundle. On the other hand, consider the mobius strip. Locally, we can write it as the product of a circle and the real line. But globally, we cannot do so due to the twist in its shape. Hence this is a non-trivial fibre bundle. If the structure group is the same as the fibre i.e, F = G then, this is known as a **principle fibre bundle**.

3.3 Connection

Now, we are ready to define the connection on a Fibre Bundle.

Vertical Subspace

The vertical subspace of the tangent space $T_u P$ is defined as: $V_u P \equiv \{X \in T_u P | \pi_* X = 0\}$

Recall that π is a map from P to B (The Base space), so its push forward is a map from T_uP to T_uB The vertical subspace consists of all those vectors in the Tangent space of P that are mapped to the null vector in the base space. We can see that only those vectors that are along the fibres will be mapped to the null vector in the base space. Hence the name vertical. We are not moving "horizontally". If we had been moving horizontally, i.e from one fibre to another , then, we couldnt have landed on the null vector in the base space. The only way that we can land on a null vector in the base space is that we move along the fibre only (since all the fibres are mapped to a single point in B, if we move only along fibre, we are not moving in B, therefore, we get a null vector.)

Connection

A connection is a smooth function of u that defines the Horizontal subspace in the following way (for each u) :

i. $T_u P = H_u P \oplus V_u P$ where $V_u P$ is defined above.

This means that we can write the tangent space as a direct sum of the vertical subspace and the horizontal subspace.

ii. $R_{q*}H_uP = H_{uq}P$

 R_{g*} is the push forward of the right action of the structure group (which is : $R_{g}u = ug$).

This is simply telling us how two horizontal subspaces along the same fibre are related. (They are on the same fibre because u and ug lie on the same fibre, g is an element of G, the structure group.)

If we have any tangent vector X in T_uP , we can clearly decompose it into two components : the vertical component (X^V) and the Horizontal component (X^H) .

Ehresmann connection

An Ehresmann connection on a principle fibre bundle is a 1-form ω that has values in the lie algebra of G such that,

i. $\omega(v_u) = \frac{dg(t)}{dt}|_{t=0}$ ii. It varies smoothly with u ($u \in P$) iii. For any $X \in T_uP$, $R_g^*\omega_{ug}(X) = \omega_{ug}(R_gX) = g^{-1}\omega_u(X)g$

This is simply another way of defining a connection. (It is defined as the projection of Tangent Space onto the vertical subspace i.e, $\omega: T_u P \to V_u P$.)

The Horizontal subspace is then simply defined as the kernel of this map :

$$H_u P = \{ X \in T_u P | \omega(X) = 0 \}$$

In terms of local co-ordinates, we have,

$$A_{\alpha} = s_{\alpha}^* \omega$$
, where

 $s_{\alpha}: U_{\alpha} \to P$, or more precisely, $s_{\alpha}: U_{\alpha} \to \pi^{-1}(U_{\alpha})$

Horizontal Lift

consider a curve $c: R \to B$ in the base manifold parametrized by t. We can lift this curve horizontally into the total space P. Call this lifted curve \tilde{c} .

The Horizontal Lift is defined as :

i. $\pi \tilde{c}(t) = c(t)$ That is when we apply the projection map on the curve in P, we get the curve in B.

ii. The tangent vectors of $\tilde{c}(t)$ lie in the Horizontal subspace. That is they have no vertical component. This concept is useful because we are only moving across fibers and not along fibers (when we say that the tangent vectors to the curve have no vertical component). We can keep moving along a single fibre if we want but that only corresponds to a single point in B.

The curve $\tilde{c}(t)$ is uniquely specified if we give its starting point. i.e we specify $\tilde{c}(0)$. This is because the condition that the curve is lifted horizontally will automatically ensure that only specific points are selected in the total space leading to the uniqueness of the lifted curve.

3.4 Holonomy

Suppose that we have a closed curve c(t) (i.e c(0) = c(T)) that is horizontally lifted. The question is that whether the lifted curve $(\tilde{c}(t))$ is also closed?

It might be closed, but in general it will not be closed. Why is this so? This is because a single point in B where c(t) lives corresponds to a whole fibre in P. So, if c(0) is mapped to $\tilde{c}(0)$, this does not mean that c(T) will also be mapped to $\tilde{c}(0)$. Rather, it can be mapped to any point on the fibre on which $\tilde{c}(0)$ is. So, in general,

$$\tilde{c}(T) \neq \tilde{c}(0)$$

However, we know that if we have to move along a fibre, it is determined by the structure group. So, these two are related by an element of the structure group. Explicitly,

$$\tilde{c}(T) = \tilde{c}(0)h$$
 where $h \in G$

This phenomena in which a closed curve in the Base Space is not necessarily lifted to a closed curve in the total space is known as **holonomy**.

The set of all of these elements h also forms a group known as the holonomy group. It is clear that this is a subgroup of the structure group.

3.5 Berry Phase as Holonomy

To recognize Berry Phase as a Holonomy, we have to look at the underlying Fibre Bundle. Consider the parameter space as the base manifold. Any state is completely characterized by the parameters R upto a phase $e^{i\phi}$. This means that we can view the total space as a fibre bundle in which the base space is the parameter space and the fibres are $e^{i\phi}$. Since they make up a Unitary group, we note that F = U(1). We also note that the structure group (which tells how to move along a fibre) is also the unitary group i.e G = U(1). So this is a principle bundle.

Recalling the gauge potential ${\cal A}_n(R)$ derived earlier in section 2 , the connection is defined as,

 $A = iA_n(R).dR = -\langle n|d|n \rangle$, where d is the exterior derivative.

The Holonomy is determined by integrating the connection over a closed loop.

 $h = \oint_C A = i \oint_C A_n(R).dR$

But recall that $\gamma(C) = \oint_C A_n(R).dR$

Hence, we can see that,

$$h = i\gamma(C) \tag{3.1}$$

Hence, it is clear that we can see the Berry Phase as a holonomy of the

connection defined on some fibre bundle.(If we know how to choose the connection!!!)

4 Spin $\frac{1}{2}$ Particle in a Magnetic Field

Consider a spin- $\frac{1}{2}$ particle that is subject to a magnetic field $\vec{B}(t)$ which is rotating adiabatically around z-axis with a frequency ω . The angle it makes with the z-axis is θ .

Then, we can write the magnetic field as,

$$\vec{B}(t) = B_0 \left(\begin{array}{c} sin(\theta) cos(\omega t) \\ sin(\theta) sin(\omega t) \\ cos(\theta) \end{array} \right)$$

The Hamiltonian of this particle is given as,

$$H = \mu \vec{B}.\vec{\sigma} \tag{4.1}$$

where, $\mu = \frac{q\hbar}{2m}$ and,

$$\sigma = \left(\begin{array}{c} \sigma_x \\ \sigma_y \\ \sigma_z \end{array}\right)$$

are the pauli matrices. Then, we get,

$$H = \mu B_0 \left(\begin{array}{c} \cos(\theta) & e^{-i\omega t} \sin(\theta) \\ e^{i\omega t} \sin(\theta) & -\cos(\theta) \end{array} \right)$$

The eigenvalue equation $H|\phi_n(t)\rangle = E_n |\phi_n(t)\rangle$, then has two solutions,

$$|\phi_{+}(t)\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ e^{i\omega t}\sin(\frac{\theta}{2}) \end{pmatrix}$$

with energy $E_+ = \mu B_0$ and,

$$|\phi_{-}(t)\rangle = \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ e^{i\omega t}\cos(\frac{\theta}{2}) \end{pmatrix}$$

with energy $E_{-} = -\mu B_{0}$

Now, to calculate the Berry phase, we can either go to the parameter space where it is given by equation (2.10) or, we can find it from (2.9) as :

$$\gamma = \int_0^T dt \ i\langle\phi|\frac{d}{dt}|\phi\rangle \tag{4.2}$$

It would be easier to use the above equation.

So, we get (with $T=\frac{2\pi}{\omega}$)

$$\gamma_{\pm} = -\pi (1 \mp \cos(\theta)) \tag{4.3}$$

Now, we look at how we can interpret this in terms of holonomy.

The base space consists of (θ, ϕ) where $\phi = \omega t$

As defined in section 3.5, the connection is $A=-\langle n|d|n\rangle$

In our case, we have,

$$|\phi_{+}(t)\rangle = \left(\begin{array}{c} \cos(\frac{\theta}{2}) \\ e^{i\phi}\sin(\frac{\theta}{2}) \end{array}\right)$$

This is a function of the parameter space (which is the base space). So, it is a 0-form.

The exterior derivative of functions or 0-forms is given as : $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$

In our case, θ is not changing, so we only have $d\phi$

Then, $d|\phi_{+}\rangle=\frac{\partial|\phi_{+}\rangle}{\partial\phi}d\phi$

$$= \left(\begin{array}{c} 0\\ ie^{i\phi}sin(\frac{\theta}{2}) \end{array}\right)d\phi$$

Then,

$$\begin{split} \langle \phi_+ | d | \phi_+ \rangle &= \frac{i}{2} (1 - \cos(\theta)) d\phi \\ \implies A &= -\langle \phi_+ | d | \phi_+ \rangle = -\frac{i}{2} (1 - \cos(\theta)) d\phi \end{split}$$

And, the holonomy is,

$$h = \oint_C A = \int_0^{2\pi} -\frac{i}{2}(1 - \cos(\theta))d\phi = -i\pi(1 - \cos(\theta))$$

Then, from (3.1), we recover the Berry phase as,

$$\gamma_+ = -\pi(1 - \cos(\theta))$$

(Similarly for γ_{-}). Which is the same as what we found before. Hence proving in this example that we can calculate Berry Phase from holonomy of some connection on a fibre bundle and that we can view Berry Phase as a Holonomy.

5 Aharonov Bohm Effect

Consider a tube in which the magnetic field inside is B and outside is 0. Also consider a box of charged particles at a distance R from this tube. The corresponding vector potential is A(r).

Now when this box is transported around the tube in a closed loop, it acquires a phase due to the presence of the vector potential even though the magnetic field outside is zero.

The Hamiltonian of a charged particle in the presence of an Electromagnetic field is, (we are taking the scalar potential (ϕ) to be zero.

$$H = \frac{1}{2m}(i\hbar\nabla - qA)^2 + V$$

Then, the solution of the Schrodinger's equation is, $\psi(r)=exp(\frac{iq}{\hbar}\int_0^r d\acute{r}~A(\acute{r})) \tilde{\psi}(r)$

where $\psi(\tilde{r})$ is a solution of the Schrödingers equation in the absence of the vector potential.

In our case of the box of particles, we can write the solution as,

$$\psi(r-R) = exp(\frac{iq}{\hbar} \int_{R}^{r} d\acute{r} A(\acute{r})) \tilde{\psi}(r-R)$$

Then,

$$\langle \psi | \nabla_{R} | \psi \rangle = \int \int \int d^{3}r \ \psi^{*} \nabla_{R} \psi \qquad (5.1)$$

$$\nabla_{R} \psi = exp(\frac{iq}{\hbar} \int_{R}^{r} d\acute{r} A(\acute{r})) \nabla_{R} \tilde{\psi} + exp(\frac{iq}{\hbar} \int_{R}^{r} d\acute{r} A(\acute{r})) \tilde{\psi} \frac{iq}{\hbar} A(R)$$

using this along with the definition of ψ in equation (5.1) and (2.14) , we get,

$$A_n(R) = \frac{-q}{\hbar} A(R) \tag{5.2}$$

And finally, we have,

$$\gamma_n(C) = \oint_C A_n(R).dR = -\frac{q}{\hbar} \oint_C A(R).dR$$
(5.3)

$$\implies \gamma_n(C) = -\frac{q}{\hbar} \int \int_S (\nabla \times A) . dS = -\frac{q}{\hbar} \int \int_S B . dS$$
(5.4)

$$\implies \gamma_n(C) = -\frac{q\phi_m}{\hbar} \tag{5.5}$$

where ϕ_m is the magnetic flux.

Now, we look at this in terms of the holonomy.

The parameter space (which is the base space) consists of R and is in fact the whole physical space.

So, the exterior derivative will act on ψ as : $d|\psi\rangle=\frac{\partial|\psi(R)\rangle}{dR}dR$

$$\langle \psi | d | \psi \rangle = \int \int \int d^3 r \ \psi^* d\psi \tag{5.6}$$

Define, $f(R) = \frac{q}{\hbar} \int_0^R A(r) dr$

Then, $d\psi = d(e^{if(R)}\tilde{\psi}(R)) = e^{if(R)}\frac{\partial\tilde{\psi}}{\partial R}dr + e^{if(R)}\tilde{\psi}i\frac{qA}{\hbar}dr$

Using this in (5.6) along with definition of connection, we get,

$$A = -\frac{iqA(R)}{\hbar}dR \tag{5.7}$$

So that,

$$h = \oint_C A = -\frac{iq}{\hbar} \oint A(R).dR = -\frac{iq}{\hbar}\phi_m \tag{5.8}$$

Comparing this with (3.1), we get,

$$\gamma_n(C) = -\frac{q\phi_m}{\hbar}$$

Which is, once again, exactly the same as we found before.

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